The Euler-Savary Formula for One-Parameter Planar Hyperbolic Motion

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March 17, 2009

Abstract

One-parameter hyperbolic planar motion was first studied by S. Yüce and N. Kuruoğlu. Moreover, they analyzed the relationships between the absolute, relative and sliding velocities of one-parameter hyperbolic planar motion as well as the related pole curves, [14]. One-parameter planar motions in the Euclidean plane $\mathbb{E}^2$ and the Euler-Savary formula in one-parameter planar motions were given by Müller, [9]. In the present article, one hyperbolic plane moving relative to two other hyperbolic planes, one moving and the other fixed, was taken into consideration and the relation between the absolute, relative and sliding velocities of this movement was obtained. In addition, a canonical relative system for one-parameter hyperbolic planar motion was defined. Euler-Savary formula, which gives the relationship between the curvature of trajectory curves, was obtained with the help of this relative system.

Mathematics Subject Classification (2000).: 53A17, 11E88.

Keywords: Kinematics, hyperbolic motion, hyperbolic numbers, Euler-Savary formula.

1 Preliminaries

Before proceeding any further, we require a definition for the set of hyperbolic number and assume the existence of any number j which has the property $j \neq \pm 1$. In terms of the standard basis $\{1, j\}$, the hyperbolic number can be written as

$$z = x + jy$$

where $j\left(j^2 = 1\right)$ is the unipotent (hyperbolic) imaginary unit and the reel numbers x and y are called the real and unipotent (or hallucinatory) parts of the hyperbolic number z, respectively, [2]-[4],[6]-[8],[10], [11]. The set of the hyperbolic numbers is

$$\mathbb{H} = \mathbb{R}\left[j\right] = \left\{ z = x + jy \,\middle|\, x, y \in, j^2 = 1 \right\}$$

In just the same way $\mathbb{C} = \mathbb{R}[i]$ are the complex numbers extended to include the imaginary i $\left(i^2 = -1\right)$ number [12], the hyperbolic numbers are the real numbers extended to include the unipotent j number.

The hyperbolic numbers are also called perplex numbers [6], split-complex numbers [1] or double numbers [1],[7],[10]. The hyperbolic number systems serve as the coordinates in the Lorentzian plane in the same way as the complex numbers serve as coordinates in the Euclidean plane. The role played by the complex numbers in Euclidean space is played by the hyperbolic number systems in the pseudo-Euclidean space, [11].

Addition and multiplication of the hyperbolic numbers are as follows:

$$(x + jy) + (u + jv) = (x + u) + j(y + v),$$
$$(x + jy)(u + jv) = (xu + yv) + j(xv + yu).$$

This multiplication is commutative, associative and distributes over addition. The hyperbolic conjugate of $z = x + jy$ is defined by $\overline{z} = x - jy$. The hyperbolic inner product is

$$\langle z, w \rangle = \operatorname{Re}(z\overline{w}) = \operatorname{Re}(\overline{z}w) = xu - yv$$

where; $z = x + jy$ and $w = u + jv$. Hyperbolic numbers z and w are hyperbolic (Lorentzian) orthogonal if $\langle z, w \rangle = 0$. Hyperbolic modulus of $z = x + jy$ is

$$\|z\|_h = \sqrt{|\langle z, z \rangle|} = \sqrt{|z\overline{z}|} = \sqrt{|x^2 - y^2|}$$

and it is the hyperbolic distance of the point z from the origin. This is the Lorentz invariant of two-dimensional special relativity and their unimodular multiplicative group (the group composed of quadratic matrices determinant of which equals to 1) is the special relativity Lorentz group, [13]. These relations have been used to extend special relativity. Furthermore, by using the functions of the hyperbolic variable, two-dimensional special relativity has been generalized, [3]. These applications make the hyperbolic numbers appropriate for physics and the application of hyperbolic numbers is similar to the application of complex numbers to the Euclidean plane geometry, [13]. Note that the points $z \neq 0$ on the lines $y = x$ are isotropic in the sense that they are nonzero vectors with $\|z\|_h = 0$. By this way, the hyperbolic distance creates Lorentzian geometry in $\mathbb{R}^2$. This is different from the usual Euclidean geometry of the complex plane, where $\|z\|_h = 0$ only if $z = 0$ in the complex plane. The set of all points in the hyperbolic plane that satisfy the equation $\|z\|_h = r > 0$ is a four-branched hyperbola of hyperbolic radius r, [11].

The hyperbolic number $z = x + jy$ can be written as follows:
While the hyperbolic number z is on H-I or H-III plane, then

$$z = \pm r \left(\cosh\varphi + j\sinh\varphi\right) = \pm re^{j\varphi},$$

While the hyperbolic number z is on H-II or H-IV plane, then

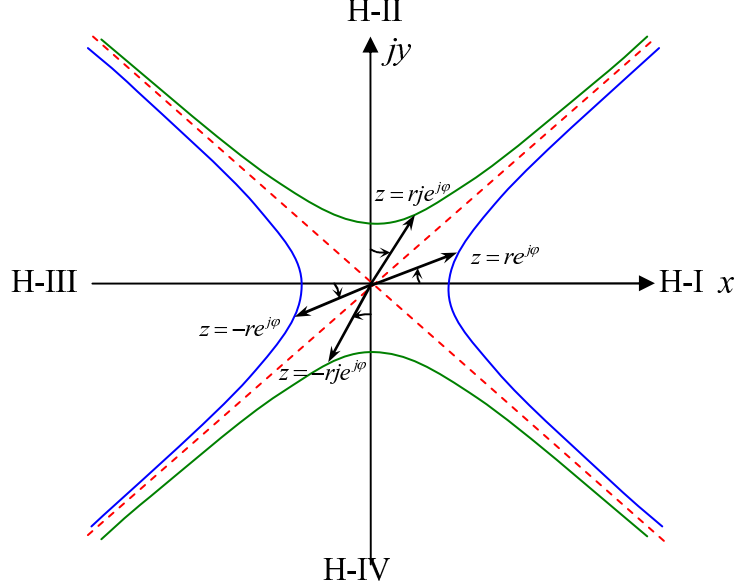$$z = \pm r \left(\sinh\varphi + j\cosh\varphi\right) = \pm rje^{j\varphi},$$

[See Figure1.1.]

Figure 1.1. Hyperbolic Plane

This formula can be derived by a power series expansion due to the fact that cosh has only even powers whereas sinh has odd powers. For all real values of the hyperbolic angle φ, the hyperbolic number $e^{j\varphi}$ has norm 1 and lies on the right branch of the unit hyperbola, [11].

A hyperbolic rotation defined by $e^{j\varphi}$ corresponds to multiplication by the matrix, [11];

$$\left[\begin{array}{cc} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{array} \right].$$

Another property of the hyperbolic inner product is

$$\left\langle ze^{j\varphi}, we^{j\varphi} \right\rangle = \left\langle z, w \right\rangle.$$

In addition, a vector multiplied by j is a hyperbolic orthogonal vector, [11]. This is similar to the role played by the multiplication $i = e^{i(\pi/2)}$ in the complex plane.

2   Planar Hyperbolic Motion

Let's consider an $\mathbb{A}$ plane which moves with regard to $\mathbb{H}$ and $\mathbb{H}'$ hyperbolic planes, first one moving and the second one fixed. Let's examine the motion of the coordinate system $\{B; \mathbf{a_1}, \mathbf{a_2}\}$ which defines hyperbolic plane $\mathbb{A}$, and hyperbolic planes $\mathbb{H}$ and $\mathbb{H}'$ with regard to the coordinate systems $\{O; \mathbf{h_1}, \mathbf{h_2}\}$ and $\{O'; \mathbf{h'_1}, \mathbf{h'_2}\}$. [See Figure 2.1. and 2.2.] If the vector $\overrightarrow{OB}$ is defined by the hyperbolic number $\mathbf{b} = b_1 + jb_2$, by applying the hyperbolic inner product, $b_1^2 - b_2^2 > 0$ or $b_1^2 - b_2^2 < 0$ can be obtained. As seen in Figure 2.1. and Figure 2.2.

respectively, the vector $\overrightarrow{OB}$ can be on the plane H-I or H-II in hyperbolic motion.
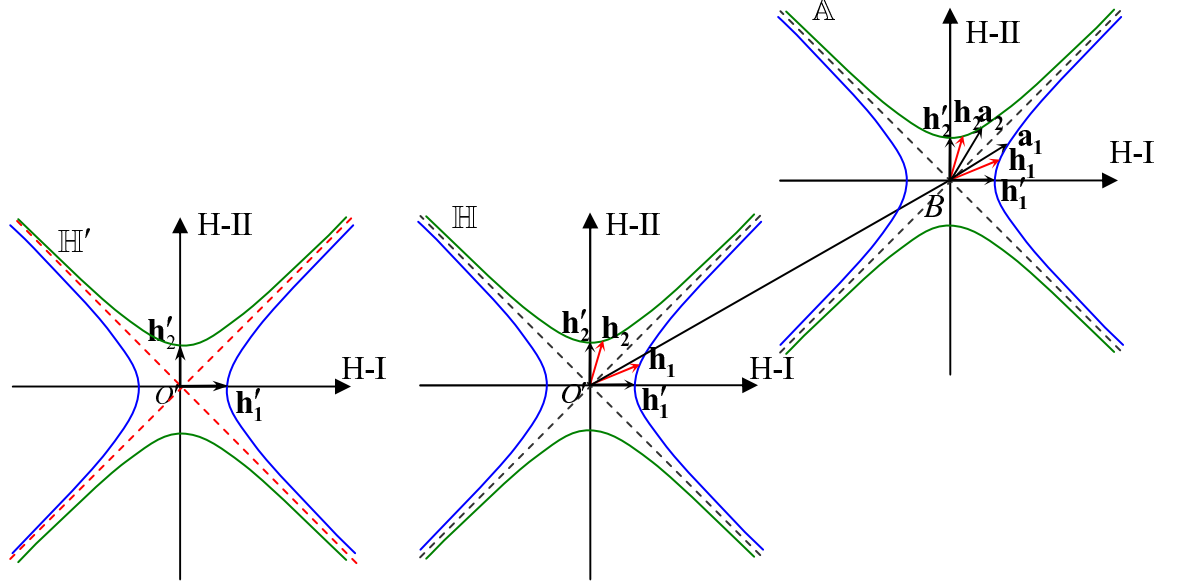

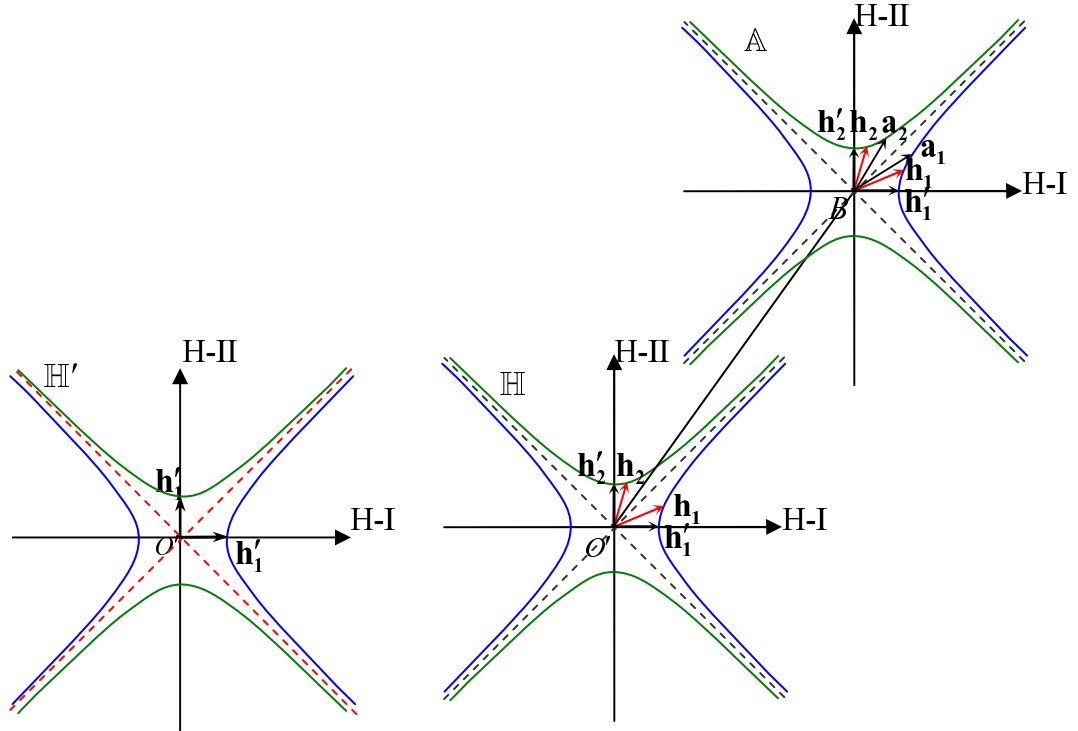Figure2.1. $\overrightarrow{OB}$ vector is on H-I plane

Figure2.2. $\overrightarrow{OB}$ vector is on H-II plane

The rotation angles of the one-parameter planar hyperbolic motion $\mathbb{A}/\mathbb{H}$ and $\mathbb{A}/\mathbb{H}'$ are φ and ψ, respectively. If the origin points of O, B and O', B are coincident, then there exists following relations;

$$\mathbf{a_1} = \cosh\varphi\,\mathbf{h_1} + \sinh\varphi\,\mathbf{h_2}$$
$$\mathbf{a_2} = \sinh\varphi\,\mathbf{h_1} + \cosh\varphi\,\mathbf{h_2}$$

and

$$\mathbf{a_1} = \cosh\psi\,\mathbf{h'_1} + \sinh\psi\,\mathbf{h'_2}$$
$$\mathbf{a_2} = \sinh\psi\,\mathbf{h'_1} + \cosh\psi\,\mathbf{h'_2}$$
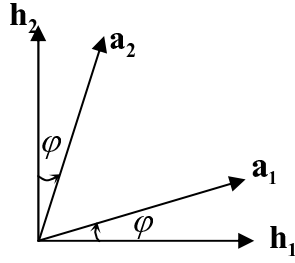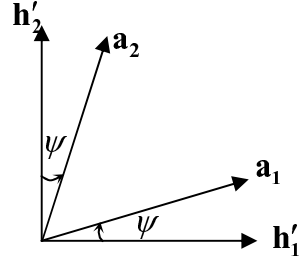
respectively [See Figure 2.3. and 2.4.].

Figure 2.3.   Figure 2.4.

If we denote the vectors $\overrightarrow{BX}$, $\overrightarrow{OB}$ and $\overrightarrow{OB'}$ with the hyperbolic numbers $\tilde{\mathbf{x}} = x_1 + jx_2$, $\mathbf{b} = b_1 + jb_2$ and $\mathbf{b'} = b'_1 + jb'_2$ on the moving coordinate system of $\mathbb{A}$, respectively; then we have

$$\mathbf{x} = (\mathbf{b} + \tilde{\mathbf{x}})\,e^{j\varphi} \qquad (2.1)$$

and

$$\mathbf{x'} = (\mathbf{b} + \tilde{\mathbf{x}})\,\mathbf{e^{j\psi}} \qquad (2.2)$$

where, the hyperbolic numbers $\mathbf{x}$ and $\mathbf{x'}$ denote the point X with respect to the coordinate systems of $\mathbb{H}$ and $\mathbb{H}'$, respectively.

Let's find the velocities of the one-parameter motion with the help of the differentiation of the equations (2.1) and (2.2). By differentiating equation (2.1), we get

$$d\mathbf{x} = (\sigma + j\tau\tilde{\mathbf{x}} + d\tilde{\mathbf{x}})\,e^{j\varphi} \qquad (2.3)$$

in which

$$\sigma = \sigma_1 + j\sigma_2 = d\mathbf{b} + j\mathbf{b}d\varphi \quad , \quad \tau = d\varphi \qquad (2.4)$$

and the relative velocity vector of X (with respect to $\mathbb{H}$) is $\mathbf{V_r} = \frac{d\mathbf{x}}{dt}$.

If we assume the differentiation of the equation (2.2),

$$d'\mathbf{x} = (\sigma' + j\tau'\tilde{\mathbf{x}} + d\tilde{\mathbf{x}})\,e^{j\psi} \qquad (2.5)$$

can be obtained along with the equation

$$\sigma' = \sigma'_1 + j\sigma'_2 = d'\mathbf{b} + j\mathbf{b'}d\psi \quad , \quad \tau' = d\psi \qquad (2.6)$$

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Also, the absolute velocity vector, that is, the velocity vector of X with respect to $\mathbb{H}'$, is $\mathbf{V_a} = \frac{d'\mathbf{x}}{dt}$.

Here, σ_i, σ_i', $(i = 1, 2)$, τ, τ' are linear differential forms of t and are called Lorentzian Pfaffian forms of one-parameter hyperbolic motion. The real parameter t represents time.

If $\mathbf{V_r} = \mathbf{0}$ and $\mathbf{V_a} = \mathbf{0}$, the point X is fixed on the hyperbolic planes $\mathbb{H}$ and $\mathbb{H}'$, respectively. Thus, the conditions of X being fixed on the $\mathbb{H}$ and $\mathbb{H}'$ planes are

$$d\tilde{\mathbf{x}} = -\sigma - j\tau\tilde{\mathbf{x}} \tag{2.7}$$

and

$$d\tilde{\mathbf{x}} = -\sigma' - j\tau'\tilde{\mathbf{x}} \tag{2.8}$$

respectively. If the equation (2.7) is substituted into equation (2.5),

$$d_f\mathbf{x} = \left[(\sigma' - \sigma) + j\left(\tau' - \tau\right)\tilde{\mathbf{x}}\right]e^{j\psi} \tag{2.9}$$

can be obtained, where the sliding velocity vector of the point X is $V_f = \frac{d_f\mathbf{x}}{dt}$. Thus, following can be easily obtained:

$$d'\mathbf{x} = d_f\mathbf{x} + d\mathbf{x} \tag{2.10}$$

Just to avoid translation, it is assumed that $\dot{\varphi} \neq 0$ and $\dot{\psi} \neq 0$. The rotation pole of the motion $\mathbb{H}/\mathbb{H}'$ is characterized by the sliding velocity P being 0. For that reason, if $d_f\mathbf{x} = \mathbf{0}$, from the equation (2.9), the pole point P of the one-parameter planar hyperbolic motion is obtained as

$$\mathbf{p} = j\frac{\sigma' - \sigma}{\tau - \tau'} \tag{2.11}$$

and if Lorentzian coordinates are preferred on the condition that $\overrightarrow{BP} = \mathbf{p} = p_1 + jp_2$, it can be written

$$p_1 = \frac{\sigma_2' - \sigma_2}{\tau - \tau'} \quad , \quad p_2 = \frac{\sigma_1' - \sigma_1}{\tau - \tau'} \tag{2.12}$$

which is given in [5].

In the $\mathbb{H}/\mathbb{H}'$ one-parameter planar hyperbolic motion, moving and fixed pole curves determine the geometric locus of the point P in $\mathbb{H}$ and $\mathbb{H}'$ planes, respectively. In other words; (P) and (P') are the representation of the moving and fixed pole curves, respectively. Also, the pole tangents can be either on the plane H-I or H-II [See Figure 2.5. and 2.6.].
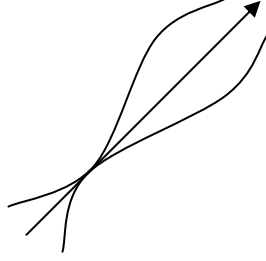
Figure 2.5.

Figure 2.6.

Let's first choose the pole tangents of the pole curves (P) and (P') on the plane H-II because the same results would be obtained by following similar operations on the plane H-I.

3 The Euler-Savary Formula for One-Parameter Planar Hyperbolic Motion

Let's choose the moving plane $\mathbb{A}$, represented by the coordinate system $\{B; \mathbf{a_1}, \mathbf{a_2}\}$, in such way to meet the following conditions:

i) The origin of the system B coincides with the instantaneous rotation pole P

ii) The axis $\{B; \mathbf{a_2}\}$ is the pole tangent, that is, it coincides with the common tangent of the pole curves (P) and (P') (on the plane H-II) [See Figure 3.1.].
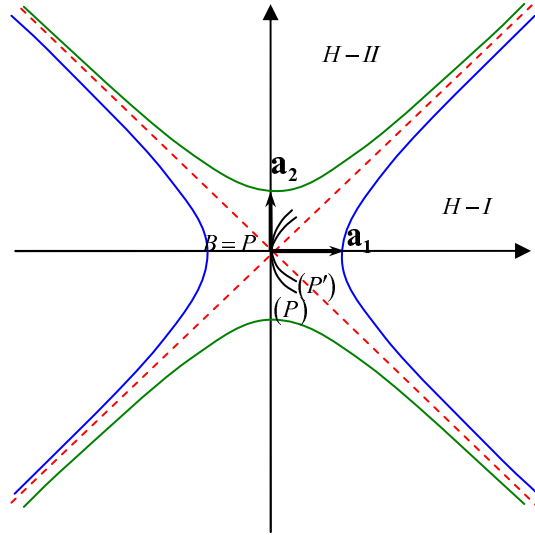

Figure 3.1.

When the condition (i) is considered: by using the equation (2.12),

$$\sigma_1 = \sigma_1' \quad , \quad \sigma_2 = \sigma_2' \tag{3.1}$$

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are obtained. From the equations (2.4) and (2.6),

$$db = (d\mathbf{b} + j\mathbf{b}d\varphi)e^{j\varphi} = \sigma e^{j\varphi}$$
$$d'\mathbf{b} = (d\mathbf{b}' + j\mathbf{b}'d\psi)e^{j\psi} = \sigma' e^{j\psi}$$

(3.2)

are found. If the equation (3.1) and the last equation are took into consideration:

$$d\mathbf{p} = d'\mathbf{p} = d\mathbf{b} = d'\mathbf{b}$$

(3.3)

is found.

Thus, the moving pole curve (P), the pole tangent of which is given, and the fixed pole curves (P') are rolling on each other without sliding.

The second condition, that is, the condition that the pole tangent coincides with $\mathbf{a_2}$, requires the coefficient of $\mathbf{a_1}$ to be zero. Here, $\sigma_1 = \sigma_1' = 0$ and $\sigma = j\sigma_2 = j\sigma_2'$ can be written. Consequently, the derivative equations of the canonical relative system $\{P; \mathbf{a_1}, \mathbf{a_2}\}$ are

$$d\mathbf{a_1} = \tau\,\mathbf{a_2} = j\tau\,e^{j\varphi}\ , \quad d\mathbf{a_2} = \tau\,\mathbf{a_1} = -\tau\,e^{j\varphi}\ , \quad d\mathbf{p} = j\sigma_2\,\mathbf{a_1} = \sigma\,e^{j\varphi} \quad (3.4)$$

and

$$d'\mathbf{a_1} = \tau'\mathbf{a_2} = j\tau'e^{j\psi}, \qquad d'a_2 = \tau'\mathbf{a_1} = j\tau'e^{j\psi}, \qquad d'\mathbf{p} = j\sigma_2'\,\mathbf{a_1} = \sigma\,e^{j\psi}$$

(3.5)

Here $\sigma = ds$ is the scalar arc element of the pole curves (P) and (P'). τ is the hyperbolic cotangent angle, that is, two neighboring tangent angles of (P). Thus, the curvature of (P) on the point P is represented by $\frac{\tau}{\sigma} = \frac{d\varphi}{ds}$.

Similarly, the curvature of the hyperbolic cotangent angle τ' -that is, the fixed pole curve (P') on the point P- is $\frac{\tau'}{\sigma} = \frac{d\psi}{ds}$.

The inverse values of these ratios

$$r = \frac{\sigma}{\tau}$$

(3.6)

and

$$r' = \frac{\sigma}{\tau'}$$

(3.7)

give the curvature radius of the pole curves (P) and (P'), respectively.

When $d\nu = \tau' - \tau$ is the infinitesimal small hyperbolic instantaneous rotation angle, the moving hyperbolic plane $\mathbb{H}$, with respect to the fixed plane $\mathbb{H}'$, rotates around the rotation pole P as much as this hyperbolic angle in the dt time scale. Thus, the hyperbolic angular velocity of the rotational motion of $\mathbb{H}$ with respect to $\mathbb{H}'$ is

$$\frac{\tau' - \tau}{dt} = \frac{d\nu}{dt} = \overset{\bullet}{\nu}\,.$$

(3.8)

From the equations (3.6), (3.7), and the last equation, the following can be written:

$$\frac{\tau' - \tau}{dt} = \frac{d\nu}{dt} = \frac{1}{r'} - \frac{1}{r}.$$

(3.9)

Let the direction of the unit tangent vector $\mathbf{a_2}$ be in the direction determined by time-based pole curves (P) and (P'). Let's choose the vector $\mathbf{a_2}$ in such

way to ensure that $\frac{ds}{dt} > 0$. In this case, $r > 0$ as the curvature center of the moving pole (P) curve is at the right side of the directed pole tangent $\{P; \mathbf{a_2}\}$. Similarly, $r' > 0$.

According to the canonical relative system, the differentiation $\mathbf{x}$- the coordinates of which are x_1, x_2- with respect to the planes $\mathbb{H}$ and $\mathbb{H}'$ are

$$d\mathbf{x} = [(\tau x_2 + dx_1) + j(\sigma_2 + \tau x_1 + dx_2)] e^{j\varphi} = (\sigma + j\tau\mathbf{x} + d\mathbf{x})e^{j\varphi} \qquad (3.10)$$

and

$$d'\mathbf{x} = [(\tau' x_2 + dx_1) + j(\sigma_2' + \tau' x_1 + dx_2)] e^{j\varphi} = (\sigma + j\tau'\mathbf{x} + d\mathbf{x})e^{j\varphi} \qquad (3.11)$$

respectively. If
$$dx_1 = \tau x_2 \quad \text{and} \quad dx_2 = -\sigma_2 - \tau x_1 \qquad (3.12)$$

then the point X is fixed on the hyperbolic plane $\mathbb{H}$. Similarly, if

$$dx_1 = \tau' x_2 \quad \text{and} \quad dx_2 = -\sigma_2' - \tau' x_1 \qquad (3.13)$$

then the point X is fixed on the hyperbolic plane $\mathbb{H}'$. Also, the sliding velocity $\mathbf{V_f}$ of the movement $\mathbb{H}/\mathbb{H}'$ corresponds to the differentiation

$$d_f\mathbf{x} = j\left(\tau' - \tau\right)(x_1 + jx_2)\, e^{j\varphi} = j\left(\tau' - \tau\right)\mathbf{x}e^{j\varphi} \qquad (3.14)$$

Now, let's examine the curvature centers of the trajectory curves drawn on their fixed plane by the points of moving planes in the motion of H/H'. In the canonical relative system, the points X, X' having the coordinates x_1, x_2 and x_1', x_2', respectively, are situated, -together with the instantaneous rotation pole P in every t moment on the instantaneous trajectory normal, which belongs to X. Moreover, this curvature center can be considered as the limit of the meeting point of the normals of the two neighboring points on the curve. Thus,

$$\begin{aligned}\overrightarrow{PX} &= x_1 + jx_2 = \mathbf{x} \\ \overrightarrow{PX'} &= x_1' + jx_2' = \mathbf{x}'\end{aligned} \qquad (3.15)$$

vectors have the same direction which passes through P. Then, for the points X and X', the equation is

$$\frac{\mathbf{x}}{\mathbf{x}'} = \frac{x_1 + jx_2}{x_1' + jx_2'} = \lambda \in \mathbb{R}. \qquad (3.16)$$

If the differential of this last equation is taken,

$$(x_1' dx_1 + x_2' dx_2 - x_1 dx_1' - x_2 dx_2') + j\left(x_1' dx_2 + x_2' dx_1 - x_1 dx_2' - x_2 dx_1'\right) = 0. \qquad (3.17)$$

If the conditions that the point X be fixed on the plane $\mathbb{H}$ and the point X' be fixed on the plane $\mathbb{H}'$ are provided, then

$$j\sigma_2\left[(x_1 + jx_2) - (x_1' + jx_2')\right] + j\left(x_1 + jx_2\right)(x_1' + jx_2')\left(\tau' - \tau\right) = 0$$

can be obtained, that is,

$$\sigma\left[\mathbf{x} - \mathbf{x}'\right] + j\mathbf{x}\mathbf{x}'(\tau' - \tau) = 0 \tag{3.18}$$

As the vectors $\overrightarrow{PX}$, $\overrightarrow{PX'}$ are on the plane H-II ,

$$\mathbf{x} = aje^{j\alpha} \tag{3.19}$$

and

$$\mathbf{x}' = a'je^{j\alpha}. \tag{3.20}$$

That is, a and a', respectively, represent the distance of the points X and X' on the plane H-II from the rotation pole P. Also, the angle α is bounded by the pole curves $\overrightarrow{PX} = \overrightarrow{PX'}$, [See Figure 3.2.]
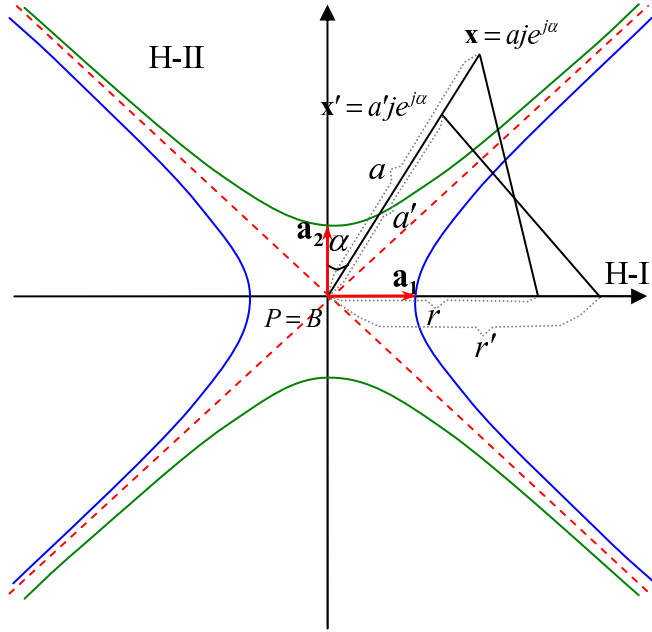

Figure 3.2.

If the equations (3.19) and (3.20) are substituted into equation (3.18), then

$$j\sigma(a - a') + jaa'e^{j\alpha}(\tau' - \tau) = 0 \tag{3.21}$$

can be obtained, and if the equation (3.9) is considered together with this last equation,

$$\frac{d\nu}{ds} = \frac{1}{r'} - \frac{1}{r} = \left(\frac{1}{a} - \frac{1}{a'}\right)e^{-j\alpha} \tag{3.22}$$

is found. Here, r and r' are the radii of curvature of the pole curves P and P', respectively. ds represents the scalar arc element and $d\nu$ represents the infinitesimal hyperbolic angle of the motion of the pole curves.

The equation (3.22) is called the Euler-Savary formula for one-parameter plane hyperbolic motion.

Consequently, the following theorem can be given.

Theorem 3.1 *Let $\mathbb{H}$ and $\mathbb{H}'$ be the moving and fixed hyperbolic planes, respectively. A point X, assumed on $\mathbb{H}$, draws a trajectory whose instantaneous center of curvature is X' on the plane $\mathbb{H}'$ in one-parameter planar motion $\mathbb{H}/\mathbb{H}'$. In the inverse motion of $\mathbb{H}/\mathbb{H}'$, a point X' assumed on $\mathbb{H}'$ draws a trajectory whose center of curvature is X on the plane $\mathbb{H}$. The relation between the points X and X' is given by the Euler-Savary formula given in the equation (3.22).*

Remark Let's choose the moving plane $\mathbb{A}$ represented by the coordinate system $\{B; \mathbf{a_1}, \mathbf{a_2}\}$ in such way to meet following conditions:

i) The origin of the system B and the instantaneous rotation pole P coincide with each other, i.e. $B = P$, [See Figure 4.1.]

ii) The axis $\{B; \mathbf{a_1}\}$ is the pole tangent, that is, it coincides with the common tangent of the pole curves (P) and (P') (on the plane H-I)
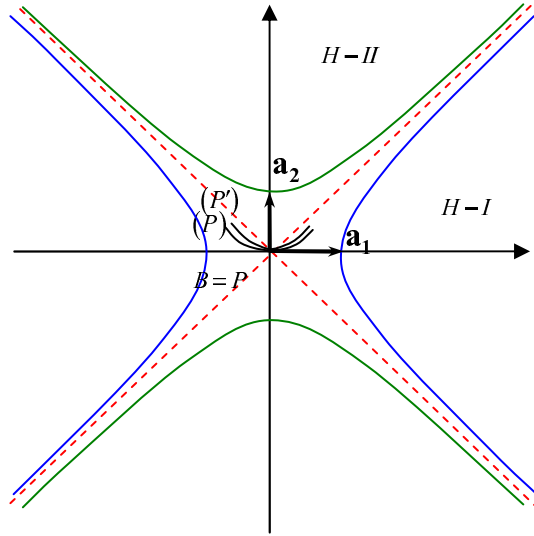

Figure 4.1.

Thus, if the operations in III. section are performed considering the conditions i) and ii), the Euler-Savary formula for one-parameter planar hyperbolic motion remains unchanged, that is, it is the same as in the equation (3.22)[See Figure 4.2.]
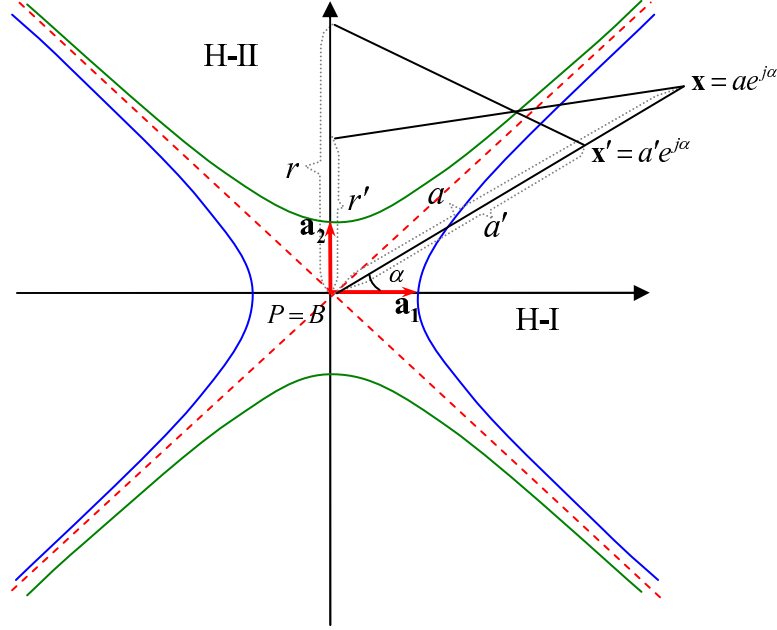
Figure 4.2.

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